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Solving Linear Fractional Programming Problem using Lu Factorization Method

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Abstract

In this paper the linear fractional programming problem is converted to a linear programming problem and for solving it, LU Factorization method is introduced. Finally, to illustrate the proposed method, it is verified by means of the numerical example and the results conclude that this method gives more reliable solution or the equal solution compared to the existing method.

Keywords: LU factorization method, fractional linear programming problem.

Introduction

The Linear Fractional Programming Problems (LFPP) (i.e. objective function has a numerator and a denominator) have attracted many researchers due to its real life application in many important fields such as health care, financial sector, production planning and hospital planning. Isbell and Marlow (1956) first identified an example of LFP problem and solved it by a sequence of linear programming problems. Charnes-Cooper (1962) have proposed a method which depends on transforming the LFPP to an equivalent linear program. Gilmore and Gomory (1963), Martos (1964), Swarup (1965), Wagner and Yuan (1968), Pandey and Punnen (2007) and Sharma et al. (1980) solved the LFP problem by various types of solution procedures based on the simplex method developed by Dantzig (1962). Bitran and Magnant (1976) have concerned duality and sensitivity analysis in LFPP . Kornbluth and Steuer (1981) discussed the feasibility region for linear fractional programming problems and multi objective linear fractional programming problems and considered its solution. Tantawy (2007, 2008) proposed two different approaches namely; a feasible direction approach and a duality approach to solve the LFP problem. Sapan Kumar das, Tarni Mandal (2015) used a New Homotopy Perturbation Method for Solving Linear Fractional Programming Problems. Furthermore, Okunev, Pavel and Johnson, Charles R., 1997 , Necessary and sufficient conditions for existence of the LU factorization of an arbitrary matrix and S. M. Chinchole, A. P. Bhadane (2014) solves the linear programming Problem using LU factorization method .

This paper is organized as follows. In section 2, Mathematical formulation for linear fractional programming problem is given. An algorithm for solving fractional linear programming problem using LU factorization is developed in section 3. In section 4, it is verified by the numerical example and lastly, we conclude our results and compare them with the original results. The last section draws some concluding remarks.

2. Mathematical Formulation

2.1 The Linear Fractional Programming Problem

$$(\text{P})\ \text{Max } z = \frac{c_1x_1+c_2x_2+c_3x_3+\cdots+c_{n-1}x_{n-1}+p}{d_1x_1+d_2x_2+d_3x_3+\cdots+d_{n-1}x_{n-1}+q}$$

Subject to $Ax \le b$, $x \ge 0$

where (i) A is an $m \times n$ matrix , $A = [\, a_{ij} \,]$ ($i = 1,\ldots,m$; $j = 1,\ldots n$)

(ii) x is a $n\times1$ column vector , b is an $m \times 1$ column vector and c ,d is the $1 \times n$ row vectors and

(iii) p and q are scalars .

2.2 LU Decomposition

An LU decomposition is a decomposition of the form A = LU where A be the square matrix, L and U are lower and upper triangular matrices (of the same size), respectively. This means that L has onlyzeros above the diagonal and U has only zeros below the diagonal.

3. An Algorithm to Solve Linear Fractional Programming Problem using Lu Factorization Method

STEP 1: Formulate the problem of (P) as

$$\text{Max } F(x) = \frac{c_1x_1+c_2x_2+c_3x_3+\cdots+c_{n-1}x_{n-1}+p}{d_1x_1+d_2x_2+d_3x_3+\cdots+d_{n-1}x_{n-1}+q}$$

Subject to

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} \le b_2,$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3,n-1}x_{n-1} \le b_3$$

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{n,n-1}x_{n-1} \leq b_n$

and $x_1, x_2, \ldots\ldots\ldots, x_{n-1} \geq 0$

Step 2 : Separate the above Linear fractional programming problem into two linear programming problem as:

Maximize F (x) = $c_1x_1 + c_2x_2 + c_3x_3 + \cdots. + c_{n-1}x_{n-1} + p$

Subject to $d_1x_1 + d_2x_2 + d_3x_3 + \cdots. + d_{n-1}x_{n-1} + q = \theta$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} \leq b_2,$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3,n-1}x_{n-1} \leq b_3$

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{n,n-1}x_{n-1} \leq b_n$

and $x_1, x_2, \ldots\ldots\ldots, x_{n-1} \geq 0$

Where θ is defined by

Minimize F (x) = $d_1x_1 + d_2x_2 + d_3x_3 + \cdots. + d_{n-1}x_{n-1} + q$

Subject to $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} \leq b_2,$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3,n-1}x_{n-1} \leq b_3$

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{n,n-1}x_{n-1} \leq b_n$

and $x_1, x_2, \ldots\ldots\ldots, x_{n-1} \geq 0$

Step 3 : Convert Minimize F(x) into Maximize G(x) so that

Maximize G(x) = $-d_1x_1 - d_2x_2 - d_3x_3 - \cdots. - d_{n-1}x_{n-1} - q$ which is the value of $-\theta$ were G = - F

Subject to $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} \leq b_2,$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3,n-1}x_{n-1} \leq b_3$

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{n,n-1}x_{n-1} \leq b_n$.

and $x_1, x_2, \ldots\ldots\ldots, x_{n-1} \geq 0$

Step 4 : Write the LPP of Step 3 in the form of less than inequalities (i.e)

To find G (x)

Subject to $d_1\, x_1 + d_2\, x_2 + d_3x_3 + \cdots + d_{n-1}\, x_{n-1} - \theta \leq -q$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} \leq b_2,$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3,n-1}x_{n-1} \leq b_3,$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{n,n-1}x_{n-1} \leq b_n$$

and $-x_1, -x_2, \ldots\ldots\ldots, -x_{n-1}, -z \leq 0$

step 5 : Check the following cases

Case I

If the number of inequalities is equal to the number of variables then LU Factorization method is applied to the system of linear equations AX = B. Go to Step 6

Case II

If the number of inequalities is less than the number of variables then add the inequalities in the system till number of inequalities equals the number of variables.(i.e)

Consider the first constraint in given LPP :

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} \leq b_2$

Choose the non zero coefficient in this inequality $a_{2j} \neq 0$ and add the inequality $a_{21}x_j \leq b_2$ in the system .Continue this process until the number of inequalities is equal to number of variables. Now, Go to step 6.

Case III

If the number of variables is less than the number of inequalities, then introduce that much number of slack variables in the appropriate inequalities and add + 1 on R.H. S. of each of that inequalities. Go to step 6.

Case IV

If there is a zero row in upper triangular matrix , then the given problem has an infeasible solution and we can stop the process.

Step 6 : Consider the system of linear equations AX = B where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots\dots & a_{1,n-1} & -1 \\ a_{21} & a_{22} & \dots\dots & a_{2,n-1} & 0 \\ . & . & & . & . \\ . & . & & . & . \\ a_{n1} & a_{n2} & & a_{n,n-1} & 0 \end{pmatrix} \quad a_{1i} = d_i \text{ , for } 1 \le i \le n-1$$

$X = [x_1 , x_{2 ,\dots\dots,} x_{n-1}, \theta]^T$ and $B = [- q , b_2, \dots \dots , b_n]^T$

(i) Find the unit lower triangular matrix L and the upper triangular matrix U such that LU = A .This will yield the equation (LU) X = B

(ii) Taking Y = UX ,solve the equation LY = B for Y where $Y = [y_1 , y_{2 ,\dots\dots,} y_{n-1}]^T$ and $B = [- q , b_2, \dots \dots , b_n]^T$

(iii) Taking the value of Y find X by solving Y = UX where $X = [x_1 , x_{2 ,\dots\dots,} x_{n-1}, \theta]^T$ by that we have calculated the value of θ

Step 7 : Consider

Maximize F (x) = $c_1x_1 + c_2x_2 + c_3x_3 + \cdots . + c_{n-1}x_{n-1} + p$

Subject to $d_1x_1 + d_2x_2 + d_3x_3 + \cdots . + d_{n-1}x_{n-1} \le \theta - q$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} \le b_2,$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3,n-1}x_{n-1} \le b_3$

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{n,n-1}x_{n-1} \le b_n$

and $x_1, x_2 ,\dots\dots\dots, x_{n-1} \ge 0$

Step 8 : To find F(x)

Subject to $-c_1 x_1 - c_2 x_2 - c_3x_3 - \cdots - c_{n-1} x_{n-1} + F(x) \le p$

$d_1x_1 + d_2x_2 + d_3x_3 + \cdots . + d_{n-1}x_{n-1} \le \theta - q$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} \le b_2,$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3,n-1}x_{n-1} \le b_3,$

$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{n,n-1}x_{n-1} \leq b_n$

and $-x_1, -x_2, \ldots\ldots\ldots, -x_{n-1}, -z \leq 0$

Step 9 : Repeat step 5 and Step 6 .By that the value of $x_1, x_2, x_3, \ldots . x_{n-1}$, F(x) is found.

Step 10 : Substituting these values of $x_1, x_2, x_3, \ldots . x_{n-1}$, in the objective function we get the solution. Also we can find the solution by dividing the value of F(x) and θ .

Note: The values of the variables found by the numerator and denominator gives the same value.

4. Numerical Example

In this section we experiments two numerical example one is a real life problem and the other is a numerical example.

Example 4.1. (Production Planning)

A company manufactures two kinds of products A and B with profit around 5 and around 3 dollar per unit, respectively. However the cost for each one unit of the above products is around 5 and around 2 dollars respectively. It is assume that a fixed cost of around 1 dollar is added to the cost function due to expected duration through the process of production. Suppose the raw material needed for manufacturing product A and B is about 3 units per pound and about 5 units per pound respectively, the supply for this raw material is restricted to about 15 pounds. Man-hours per unit for the product A is about 5 hour and product B is about 2 hour per unit for manufacturing but total Man-hour available is about 10 hour daily. Determine how many products A and B should be manufactured in order to maximize the total profit.

We formulate this real life production planning problem in to LFP problem as follows:

Max $F(x) = \frac{5x_1 + 3x_2}{5x_1 + 2x_2 + 1}$

Subject to the constraints

$$3x_1 + 5x_2 \leq 15$$

$$5x_1 + 2x_2 \leq 10$$

$x_1, x_2 \geq 0$

Separate the LFP problem as

Max $F(x) = 5x_1 + 3x_2$

Subject to the constraints $5x_1 + 2x_2 + 1 \leq \theta$

$3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$

Subject to the constraints $5x_1 + 2x_2 + 1 \leq \theta$
$3x_1 + 5x_2 \leq 15$
$5x_1 + 2x_2 \leq 10$
$x_1, x_2 \geq 0$

Using Step 3 and Step 4 the LPP can be written as

Max G(x)

Subject to $55\, x_1 + 2x_2 - \theta \leq -1$
$3x_1 + 5x_2 \leq 15$
$5x_1 + 2x_2 \leq 10$
$- x_1, - x_2, -z \leq 0$

Applying Step 5and step 6, the initial iteration is

$y_1 = -1$, $y_2 = 78/5$, $y_3 = 11$

Again on simplification, we get the final iteration as

$x_1 = 1.052$, $x_2 = 2.368$, $\theta = 11$

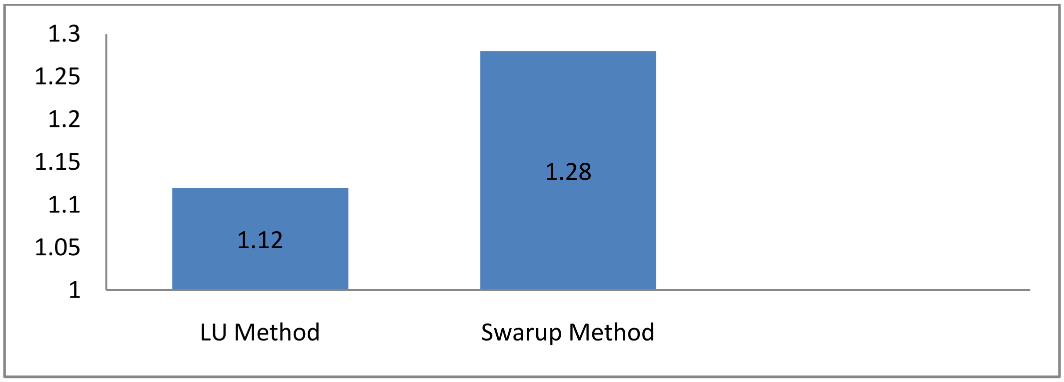
Now Applying Step 7 ,

Max $F(x) = 5x_1 + 3x_2$
Subject to the constraints
$5x_1 + 2x_2 \leq 10$
$3x_1 + 5x_2 \leq 15$
$5x_1 + 2x_2 \leq 10$, $x_1, x_2 \geq 0$

Using Step 4, Let us write the LPP as
Find F(x)
Subject to $-5\, x_1 - 2x_2 + F(x) \leq 0$
$5x_1 + 2x_2 + x_3 \leq 11$
$3x_1 + 5x_2 \leq 15$
$5x_1 + 2x_2 \leq 10$,- $x_1, - x_2, x_3 , -z \leq 0$

Using step 5,6 the initial iteration is $y_1 = 0$, $y_2 = 11$, $y_3 = 251/5$,$y_4 = -1$ By that we can find X as $x_1 = 1.052$, $x_2 = 2.368$,F(x) = 12.368, $x_4 = 0$ Substituting the value of X in the objective function we get Max $F(x) = 1.12$ By comparing the result of proposed method with Swarup method, we conclude that our method is more reliable, because : By our Proposed Method F (x) = 1.12 But by Swarup Method F (x) = 1.28

The graphical representation is as follows:



Example 4.2

Max $F(x) = \frac{5x_1 + 2x_2}{x_1 + 8x_2 + 1}$

Subject to the constraints

$$3x_1 + x_2 \geq 1$$

$$x_1, x_2 \geq 0$$

Separate the LFP problem as

Max $F(x) = 5x_1 + 2x_2$

Subject to the constraints $x_1 + 8x_2 + 1 \leq \theta$

$$3x_1 + x_2 \geq 1$$

$$x_1, x_2 \geq 0$$

Where θ is defined by Minimize F(x) = $x_1 + 8x_2 + 1$

Subject to $3x_1 + x_2 \geq 1$

$$x_1, x_2 \geq 0$$

Using step 3,4,5 and 6 the initial iteration is $y_1 = -1$, $y_2 = -4$, $y_3 = -27/23$ Again On simplification , we get the final iteration as $x_1 = 0$, $x_2 = 1$, $\theta = 9$

Now Applying Step 7 ,

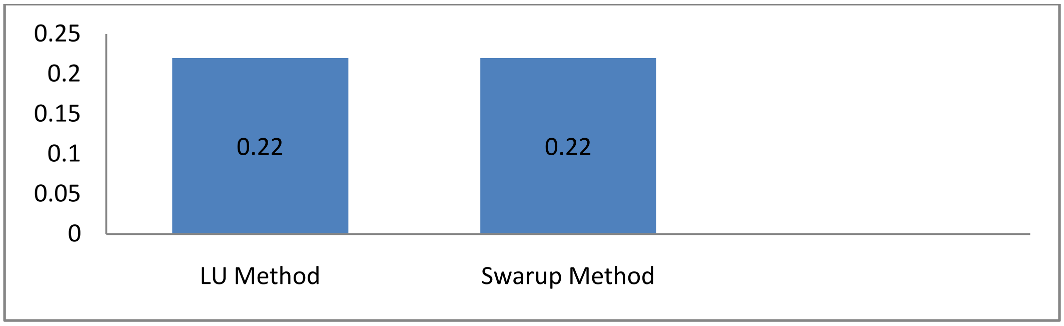
Max $F(x) = 5x_1 + 2x_2$

Subject to the constraints $x_1 + 8x_2 \leq 8$

$-3x_1 - x_2 \leq -1,$

$, x_1, x_2 \geq 0$

Using Step 4,5 and 6 the initial iteration is $y_1 = 0$, $y_2 = 8$, $y_3 = -23/19$ By that we can find X as $x_1 = 0$, $x_2 = 1$, F(x) = 2 Substituting the value of X in the objective function we get Max $F(x) = 0.22$. Again we conclude that by our Proposed Method F (x) = 0.22 and by Swarup Method F (x) = 0.22 The graphical representation is as follows:



5. Conclusion

In this paper, the linear fractional programming problem is considered because the optimal solution of LFPP is often more preferable and attractive than the LP problems because of higher efficiency. Here,we convert the LFPP to the equivalent Linear programming problems and solve by using LU factorization method . By operating LU factorization method , time can be saved using fewer calculations as compared to the Swarup method. We also present a graphical comparison of the LU Factorization method with the existing Swarup method by showing the reliability and applicability of our algorithm.

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